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Applicability of the Lewis and Aboav–Weaire laws to 2D and 3D cellular structures based on Poisson partitions

M A Fortes

Departamento de Engenharia de Materiais, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal

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Abstract. Two- and three-dimensional networks of a columnar type are described, which result from partitions based on Poisson point distributions. The metric and topological properties of such laminated Poisson networks are derived and the applicability of the Lewis and Aboav–Weaire laws to them is tested. The 2D Poisson network contains cells with $i \geqslant 4$ (i is the number of sides) and both laws are obeyed. The 3D Poisson networks are of various types and have $F \geqslant 6$ (F is the number of faces in a cell) and $\bar{F} = 14$. In one particular type of 3D Poisson network the two laws are again exactly obeyed. In another type, the laws show large deviations at low F but are asymptotically obeyed when F tends to infinite.

1. Introduction

Cellular structures, ranging from soap froths to polycrystals and solid foams, have been the subject of much interest in recent years (e.g. Weaire and Rivier 1984, Glazier and Weaire 1992, Gibson and Ashby 1988). Attempts have been made to establish the general topological and geometrical properties of such structures which may help in explaining and predicting their peculiar properties and behaviour. Most studies have concentrated on 2D cellular structures which may be represented by random planar networks, usually with trivalent vertices (i.e. three edges at each vertex). Excluding those properties that are a direct consequence of Euler's theorem, the best documented general topological property of such 2D networks is the Aboav–Weaire law for the average number m_i of sides (edges) in cells adjacent to cells with i sides (i cells). This quantity is approximately linear in $1/i$ in the form (Aboav 1970, 1980, Weaire 1974)

$$im_i = (6-a)i + 6a + \mu_2 \tag{1}$$

where a is a constant for each network and μ_2 is the second moment of the distribution $g(i)$ of the number of sides. The fact that only one parameter, a, appears in equation (1) is a consequence of the following identity, first derived by Weaire (1974),

$$\langle im_i \rangle = \langle i^2 \rangle = \langle i \rangle^2 + \mu_2 \tag{2}$$

where $\bar{i} = \langle i \rangle = 6$ is the average value of i in trivalent 2D networks. Equation (1) can be predicted from maximum entropy arguments (Peshkin *et al* 1991) but is, in general, only approximate in actual networks. The exact dependence of m_i on i was recently derived for a few types of random trivalent networks, which shows that the linear relation (1) is indeed only approximately obeyed. In the network analysed by Godrèche *et al* (1992), i varies

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in the interval $[3, \infty]$. An equation for im_i was derived which contains a nonlinear extra term in $(i+1)^{-1}$. The family of networks analysed by Le Caer (1991) and Delannay *et al* (1992) depends on a parameter, p. The networks all have i in the interval $[4, 8]$. The Aboav–Weaire law is not exactly obeyed, except for one particular value of the parameter p. This is apparently the only available non-trivial example of network for which the Aboav–Weaire law is obeyed exactly. In this paper we give another example for a network with i in the interval $[4, \infty)$. Exact results of im_i in larger intervals of i have also been given by Le Caer and Delannay (1993) for 2D networks associated with tilings of triangles; again Aboav–Weaire law is only approximately followed.

There has been some speculation on whether Aboav–Weaire law is applicable to 3D random networks (with tetravalent vertices) to describe the correlation between the number of faces F in adjacent cells. The 3D form of (1) is

$$Fm_F = (\bar{F} - a)F + \bar{F}a + \mu_2 \tag{3}$$

where $\bar{F}$ and μ_2, respectively, are the average value and the second moment of the distribution of F, and m_F is the average number of faces in cells face-adjacent to F cells. The dependence on a single parameter a is a consequence of the following identity, analogous to equation (2):

$$\langle Fm_F \rangle = \bar{F}^2 + \mu_2. \tag{4}$$

A few examples have been reported (Fortes 1989, 1993) of 3D networks to which (3) is approximately applicable, in particular the 3D Voronoi network. But the 3D Aboav–Weaire law has not been as thoroughly tested as the 2D version. In this paper we analyse special types of 3D tetravalent networks that confirm the applicability of (3). In one case the law is exactly followed.

Another general relation of a different nature that has been claimed to apply to cellular structures is the Lewis law (Lewis 1928, 1931). This is an empirical relation which states that the average area (volume) of cells is approximately linear in the number of topological elements (edges in 2D, faces in 3D). Rivier and Lissowski (1982) have shown that the Lewis law can be derived as a consequence of the maximum entropy principle. A similar law, known as Desch law (see Rivier 1985), has been suggested for the relation between the perimeter (2D) or surface area (3D) and the number of topological neighbours.

We will check the applicability of the Lewis law to the class of 2D and 3D networks that will be analysed in this paper. The law is exactly followed in some networks and approximately followed in others. Interestingly, we find that the Lewis law is exact when Aboav–Weaire law is exact.

2. Description of networks

The 2D trivalent network (to be termed 2D laminated Poisson network) is generated as follows (see figure 1). We take a family of parallel lines in the x direction in the plane. The distance between lines does not affect the topological properties of the network although it affects the geometry. When representing the networks and also for the purpose of obtaining cell areas, we take that distance as uniform (d_0) as in figure 1(a). Each region (or column) between successive lines is divided into cells (figure 1(b)). The division of a column is based on a Poisson point distribution in the x direction: an edge in the y direction is taken through each Poisson point (P-point). The distributions in the various columns are

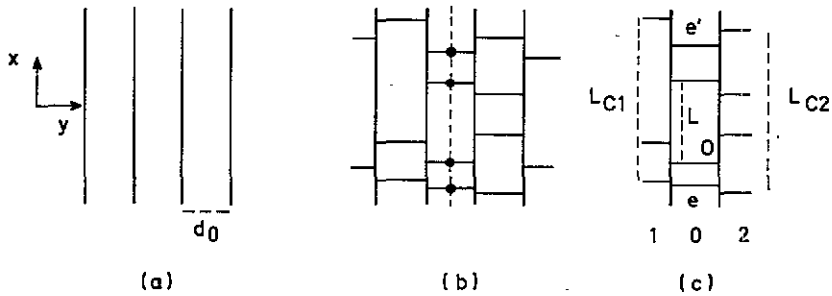

Figure 1. Two-dimensional Poisson network. Parallel columns in the x direction (a) are divided into cells by edges through the points of independent Poisson distributions (b). For a cell 0 of height L, the lengths L_{c_1} and L_{c_2} are the cover lengths in adjacent columns 1 and 2. Edges e and e' are of cells base-adjacent to cell 0.

independent but all have the same density which we take to be one P-point per unit length. The height of a cell, L, is the distance between adjacent P-points (figure 1(c)). Its average value is then unity. Networks based on Poisson distributions of non-uniform density can also be defined and analysed but will not be considered in this paper.

The 3D tetravalent networks (3D laminated Poisson networks) are generated by a similar process (figure 2). We take an arbitrary 2D trivalent network (base network) on the xy plane (figure 2(a)) and take vertical (parallel to z) planes through each edge which divide the 3D space into prismatic columns. In each column we consider a Poisson distribution of unit density in the z direction and divide the column into cells by planes perpendicular to z through each P-point (figure 2(b)). The height of a cell, L, is the distance between adjacent P-points in the associated distribution (figure 2(c)). The average cell height is unity. Two particular types of these 3D networks will be analysed in detail. One is based on the hexagonal network. The other is based on the 2D laminated Poisson network. The first will be termed hexagonal laminated Poisson network and the other the doubly laminated Poisson network.

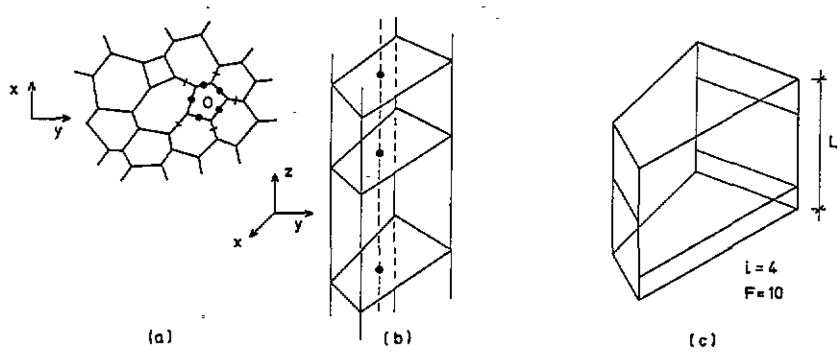

Figure 2. Three-dimensional Poisson networks. The 3D networks are based on arbitrary planar (xy) networks (a) which define the columnar structure in the z direction (b). The cell in (c) has $i = 4$ and $F = 10$ and height L.

3. Basic equations

There are various properties of the Poisson distribution that will be used to analyse the topological and geometrical properties of the 2D and 3D Poisson networks previously described. In this section we give all the relevant equations. Consider a Poisson point distribution in a straight line, of unit density. The distribution can be realized by placing M random points on a segment of length M and letting M tend to infinite. The probability density $p(L)$ of a Poisson segment of length L is

$$p(L) = e^{-L} \tag{5}$$

such that $p(L)\,dL$ is the probability of a segment length in the interval $L, L + dL$ (for short L, dL). Note that the average length, $\bar{L}$, is taken to be unity. If we take n segments (adjacent or not) the probability density for the sum, L_t, of their lengths is

$$\pi(n, L_t) = \frac{L_t^{n-1} \exp(-L_t)}{(n-1)!}. \tag{6}$$

If we take an independent segment (not a P-segment) in the distribution, of length L_0, the probability that it contains exactly n P-points is

$$P(n, L_0) = \frac{L_0^n \exp(-L_0)}{n!}. \tag{7}$$

These probabilities are frequently used to define the Poisson point distribution.

The expected (or average) number of P-segments in a segment L_0 is

$$\langle n \rangle_0 = L_0. \tag{8}$$

This result applies even if the origin of L_0 is at a P-point. Equations (5)–(8) are standard properties of the 1D Poisson point distribution.

In the following we consider a number N of Poisson distributions (numbered $1, \ldots, N$) overlapping another distribution, which we denote by 0-distribution, as in the example of figure 3(a). All distributions have unit density. We wish to find the probability density of P-segments of length L, dL in the 0-distribution containing $n_1 \ldots n_N$ points of distributions $1, \ldots, N$, respectively. The probability density $\varphi(L; n_1 \ldots n_N)$ is proportional to the product $p(L)P(n_1, L) \ldots P(n_N, L)$. Upon normalization we obtain, using (5) and (7),

$$\varphi(L; n_1 \ldots n_N) = \varphi(L, I) = \frac{(N+1)^{I+1}}{I!} L^I \exp[-(N+1)L] \tag{9a}$$

where

$$I = \sum_{i=1}^{N} n_i. \tag{9b}$$

The distribution of L is therefore the same for all decompositions of I in N terms. Therefore we write $\varphi(L, I)$.

It is easily shown that a given I can result from N^I decompositions of type (9b) (for $N = 2$, $I = 3$, for example, $n_1 = 1$, $n_2 = 2$ is taken as distinct from $n_1 = 2$, $n_2 = 1$). The following identities will be useful later:

$$\sum \frac{I!}{n_1! \ldots n_N!} = N^I \tag{10}$$

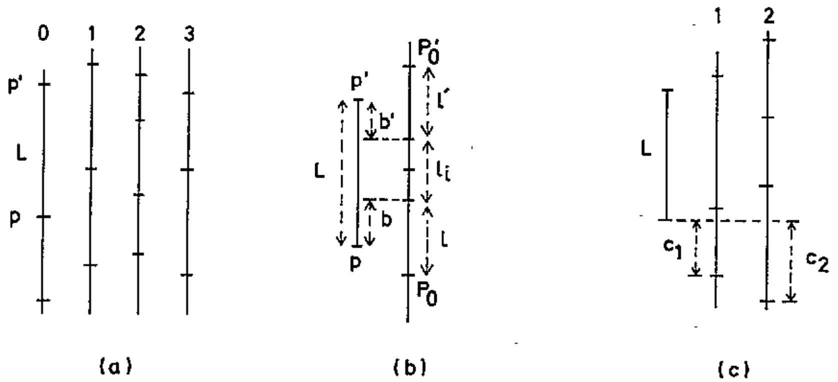

Figure 3. (a) Distributions 1, 2, 3 overlapping a reference distribution 0. (b) Segment PP' of length L and distribution 0 of Poisson points. Points P_0 and P'_0 define the cover length L_c in 0 of segment L. (c) Cover lengths of segment L in two distributions 1 and 2. The two cover segments partly overlap. The non-overlap length at the bottom is $|c_2 - c_1|$.

where the sum is for all the N^I decompositions of I and

$$\sum_{I=0}^{\infty} \frac{N^I}{(N+1)^{I+1}} = 1. \tag{11}$$

The fraction, $(n_1 \ldots n_N)$, of segments (with any L) of the 0-distribution that correspond to a particular decomposition $n_1 \ldots n_N$ (i.e. that contains n_1 P-points of distribution 1, etc) is

$$(n_1 \ldots n_N) = \int_0^\infty p(L) P(n_1, L) \ldots P(n_N, L) \, \mathrm{d}L = \frac{I!}{(N+1)^{I+1}} \frac{1}{n_1! \ldots n_N!}. \tag{12}$$

The sum of the quantities $(n_1 \ldots n_N)$ for all decompositions of all Is (in the interval $(0, \infty)$) is unity. This is easily proved from identities (10) and (11). The average length of segments of the 0-distribution with fixed I can be obtained from $\varphi(L, I)$ (equation (9*a*)). The result is

$$\langle L \rangle_I = \frac{I+1}{N+1}. \tag{13}$$

Next we consider a segment of length L randomly located in the 0-distribution and of extremities P and P'. This segment can be a Poisson segment of another Poisson distribution. The cover length, L_c, of L is defined as the length of the segment $P_0 P'_0$ (figure 3(b)), P_0 and P'_0 being the points of the 0-distribution that immediately follow the extremities P and P' of L, and are outside L. We wish to find the distribution of the cover lengths, L_c, of segments L such that there are n points of the 0-distribution in L. We show that the probability density of L_c, for given L and n, is given by

$$\chi(L_c, L) = (L_c - L) \exp[-(L_c - L)] \qquad L_c \geqslant L \tag{14}$$

and is therefore independent of n. To derive (14) we refer to figure 3(b). The lengths of the extreme P-segments are ℓ, ℓ' and the distances between the extreme points P and P' of

L to P_0 and P_0' are $\ell - b$ and $\ell - b'$, respectively. The total length between P_0 and P_0' is $\ell + \ell' + \ell_i$. We then have

$$L = \ell_i + b + b'$$
$$L_c = \ell_i + \ell + \ell'. \tag{15}$$

It is necessary that $\ell > b$ and $\ell' > b'$. The interval of ℓ for fixed b is then

$$b \leqslant \ell \leqslant L_c - L + b. \tag{16}$$

A cover length in the interval L_c, $\mathrm{d}L_c$ with fixed b and ℓ_i can result from $\ell, \mathrm{d}\ell$ with probability $p(\ell)\,\mathrm{d}\ell$ and ℓ', $\mathrm{d}\ell'$ with probability $p(L_c - \ell_i - \ell)\,\mathrm{d}L_c$. Then the probability density of L_c for fixed b and ℓ_i is

$$\chi(L_c, L; \ell_i, b) = k\int_b^{L_c-L+b} p(\ell)p(L_c - \ell_i - \ell)\,\mathrm{d}\ell = k(L_c - L)\exp[-(L - \ell_i)] \tag{17}$$

where k is a normalization factor. Note that there is no dependence on b. The factor k is determined from

$$\int_L^{\infty} \chi(L_c, L; \ell_i, b)\,\mathrm{d}L_c = 1 \tag{18}$$

leading to the result expressed in equation (14). The probability density χ is also independent of ℓ_i.

The previous derivation is not applicable for $n = 0$ (no points inside L) because ℓ_i cannot be defined. The derivation can easily be modified and leads to the same distribution, i.e., to the same χ.

The average value of $(L_c - L)$ for fixed L is 2, independent of L,

$$\langle (L_c - L) \rangle = 2. \tag{19}$$

Finally, we consider, for a given segment L, the cover lengths L_{c_1}, L_{c_2} in two independent Poisson distributions. These cover lengths partly overlap (figure 3(c)). The non-overlap length L_{n0} has the average value $2\langle |c_2 - c_1| \rangle$ where the cs are defined in figure 3(c). The probability density of c_i is $\exp(-c_i)$. This leads to the value 2 for the expected length of non-overlapped regions in the two distributions:

$$\langle L_{n0} \rangle = 2. \tag{20}$$

4. Analysis of laminated Poisson networks

4.1. Distribution of cells in topological classes

4.1.1. Two-dimensional networks. The number of sides, i, of a cell in a 2D Poisson network is

$$i = 4 + I \qquad I = n_1 + n_2 \tag{21}$$

where n_1, n_2 are the numbers of additional vertices contributed by the cells in adjacent columns ($n_1 = 1$ and $n_2 = 2$ for cell 0 in figure 1(c)). The fraction, $g(i)$, of i cells is then

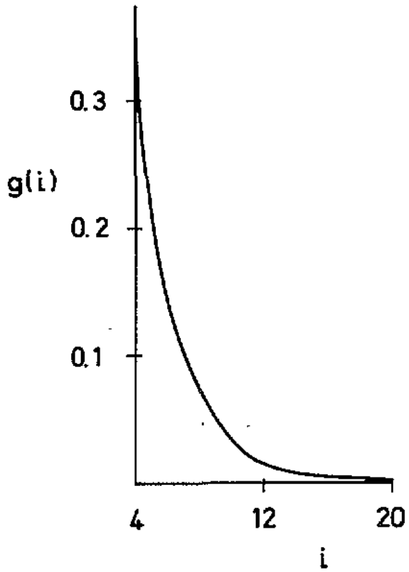

Figure 4. The fraction $g(i)$ of cells with i sides in the 2D Poisson network.

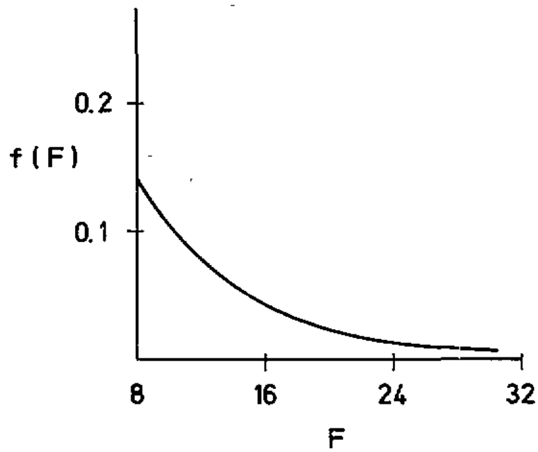

Figure 5. The fraction $f(F)$ of cells with F faces in 3D Poisson network based on the hexagonal network.

the sum of the fractions (n_1n_2) for all combinations that lead to $I = i - 4$. The quantity (n_1n_2) is given by (12) with $N = 2$. Using (10), we find

$$g(i) = \sum_I (n_1n_2) = \frac{2^I}{3^{I+1}} = \frac{1}{3}\left(\frac{2}{3}\right)^{i-4}. \tag{22}$$

The distribution is plotted in figure 4. Note that the fraction of cells with i sides in a column is also $g(i)$.

The following results can be obtained with the aid of identities given in the appendix:

$$\sum_{i=4}^{\infty} g(i) = 1 \qquad \langle i \rangle = 6 \qquad \langle i^2 \rangle = 42 \qquad \mu_2 = 6. \tag{23}$$

4.1.2. Three-dimensional networks. In a 3D laminated Poisson network based on a 2D network with an arbitrary distribution $g(i)$ of the number of edges ($i \geqslant 3$), the number of faces in a cell with an i-base is

$$F = i + 2 + I \qquad I = n_1 + \cdots + n_i \tag{24}$$

where i is the number of columns adjacent to the column where the cell is located. The cell has a number, n_k, of additional lateral faces which result from the cells in the adjacent k-column (figure 2(c)).

For fixed i, the fraction $f_i(I)$ of cells with I is obtained by summing the fractions $(n_1 \ldots n_i)$ for all combinations compatible with I. The result is

$$f_i(I) = \frac{1}{i+1}\left(\frac{i}{i+1}\right)^I. \tag{25}$$

The average number of faces $\bar{F}_i$ for fixed i is easily found to be

$$\bar{F}_i = 2i + 2. \tag{26}$$

The fraction f_{iF} of cells with fixed i and F is

$$f_{iF} = \frac{g(i)}{i+1}\left(\frac{i}{i+1}\right)^{F-i-2} \qquad 3 \leqslant i \leqslant F-2. \tag{27}$$

For fixed F the fraction, $f_F(i)$, of cells with i-bases is

$$f_F(i) = \frac{f_{iF}}{\sum_{i=4}^{F-2} f_{iF}} \qquad i \geqslant 3. \tag{28}$$

Finally the fraction $f(F)$ of F-cells is

$$f(F) = \sum_{i=3}^{F-2} f_{iF} \qquad F \geqslant 5. \tag{29}$$

It is possible to calculate $\bar{F}$ and $\langle F^2 \rangle$ with the help of identities given in the appendix:

$$\bar{F} = \langle F \rangle = 14 \qquad \langle F^2 \rangle = 58 + 5\langle i^2 \rangle \qquad \mu_2 = 42 + \mu_2^{(i)} \tag{30}$$

where $\mu_2^{(i)}$ is the second moment for the 2D base network. All 3D Poisson networks have $\bar{F} = 14$.

We apply these results to two types of 3D laminated Poisson networks. In the first type the 2D base network is the hexagonal network. Then $i = 6$ in all cells ($g(6) = 1$). We have

$$f(F) = f_{6F} = \tfrac{1}{7}(\tfrac{6}{7})^{F-8} \qquad F \geqslant 8. \tag{31}$$

The distribution is plotted in figure 5. It has $\bar{F} = 14$ and $\mu_2 = 42$. The fraction of F-cells in each column is also $f(F)$.

The other 3D network is based on the 2D laminated Poisson network. It will be termed a doubly laminated Poisson network. The number of faces in a cell is

$$F = i + 2 + I = 6 + I_0 + I \tag{32}$$

where I_0 is the number of extra lateral edges in the base and I the number of extra lateral faces. The fractions $g(i)$ are given by (22). Using the previous equations we obtain

$$f_{iF} = \frac{1}{3}\left(\frac{2}{3}\right)^{i-4} \frac{1}{i+1}\left(\frac{i}{i+1}\right)^{F-i-2} \qquad 4 \leqslant i \leqslant F-2 \tag{33}$$

and for the fraction of F-cells

$$f(F) = \sum_{i=4}^{F-2} \frac{1}{3(i+1)}\left(\frac{2}{3}\right)^{i-4}\left(\frac{i}{i+1}\right)^{F-i-2}. \tag{34}$$

The distribution is plotted in figure 6(a). It has $\bar{F} = 14$ and $\mu_2 = 72$.

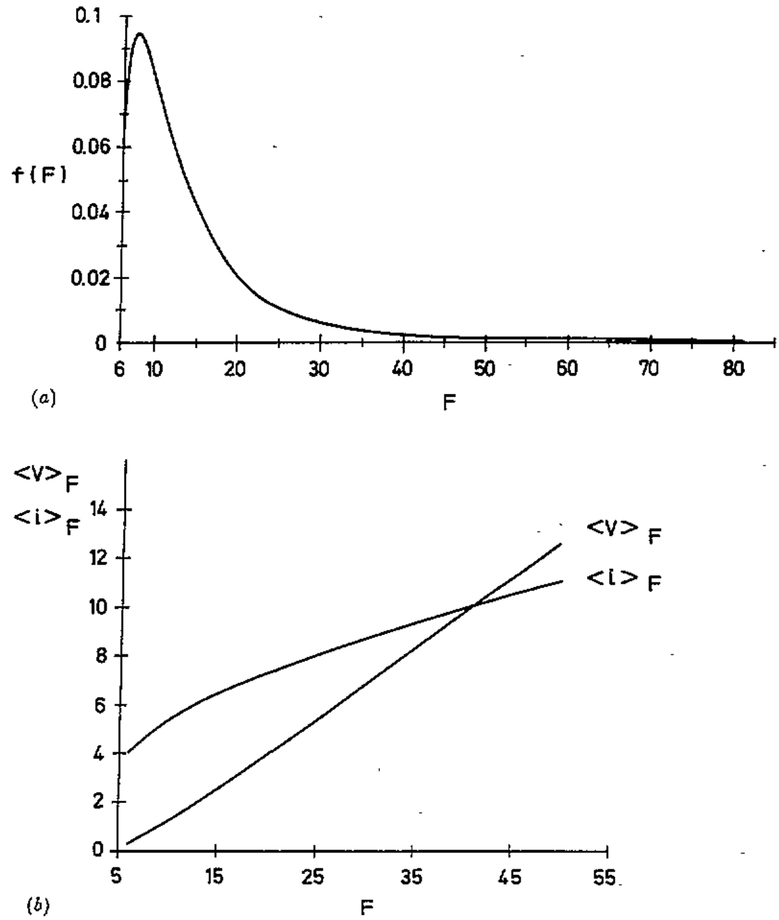

Figure 6. 3D doubly laminated Poisson network: (a) the fraction $f(F)$ of cells with F faces; (b) the average number of lateral faces, $\langle i\rangle_F$ in cells with fixed F and the quantity $\langle V\rangle_F = \langle L_b L_l\rangle_F$, proportional to the average volume of cells with F faces.

4.2. Lewis law

4.2.1. Two-dimensional networks. In the 2D laminated Poisson network the area of an i cell of height L is $A = Ld_0$ where d_0 is the width of the columns (or its average value if the width is non-uniform). The average L of cells with fixed i is, from (13) with $N = 2$ and $I = i - 4$,

$$\langle L\rangle_i = \frac{i-3}{3}. \tag{35}$$

The area of i cells is then linear in i and Lewis law is exact. A linear relation in i also holds for the average perimeter, $\langle 2(L + d_0)\rangle$, of the cells.

The distribution $\varphi(L, i)$ of lengths, L, of cells with fixed i is given by (9a) with $N = 2$, $I = i - 4$, and can also be used to obtain (35).

4.2.2. Three-dimensional networks. In the 3D hexagonal laminated Poisson network the volume of a cell of height L is $V = A_0L$ where A_0 is the area of the hexagonal base. For fixed F, the average height is given by (13), leading to

$$\langle L\rangle_F = \frac{F-7}{7}. \tag{36}$$

Lewis law is again exact. The average surface area of F cells is also linear in F. The distribution $\varphi(L, F)$ of heights, L, of cells with fixed F is given by (9a) with $N = 6$ and $I = F - 8$ and can also be used to obtain (36).

In the 3D doubly laminated Poisson network the volume of a cell is $V = d_0 L_b L_l$. The average height L_b of the bases with given $I_0 = i - 4$ is

$$\langle L_b\rangle_i = \frac{i-3}{3} = \frac{I_0+1}{3} \tag{37}$$

and the average height L_l of the cells with given i and given $I = F - i - 2$ is

$$\langle L_l\rangle_{i,F} = \frac{I+1}{I_0+1}. \tag{38}$$

For fixed F the fraction of cells with i bases, $f_F(i)$, is given by (28). The average value of $L_b L_l$ for fixed F is

$$\langle L_b L_l\rangle_F = \sum_{i=1}^{F-2} f_F(i)\left(\frac{I+1}{3}\right). \tag{39}$$

This can be written in the form

$$\langle L_b L_l\rangle_F = \tfrac{1}{3}(F - 1 - \langle i\rangle_F) \tag{40}$$

where $\langle i\rangle_F$ is the average value of i for fixed F:

$$\langle i\rangle_F = \sum_{i=4}^{F-2} i f_F(i). \tag{41}$$

This quantity is plotted in figure 6(b). It is apparent that cells with smaller F have smaller i, but $\langle i\rangle_F$ is not linear in F.

Figure 6(b) also shows a plot of the quantity $\langle V\rangle_F = \langle L_b L_l\rangle_F$, proportional to the volume of a cell, as a function of F. Lewis law is not exactly followed. The deviation from linearity (which is related to the nonlinear dependence of $\langle i\rangle_F$ on F) is noticeable at low F, but the linear relation holds for large F. The best fit straight line depends on the interval of F where the fitting is done. For F in the interval [6, 50] the best linear relation (least-squares fit) is

$$\langle L_b L_l\rangle_F = -1.697 + 0.284F. \tag{42}$$

Note that the average value of the quantities $\langle L_b L_l\rangle_F$ for all F is unity. The average surface area of cells with fixed F is also not linear in F; we have not calculated its variation with F.

4.3. Aboav–Weaire law

We wish to find the expected number of topological elements (edges in 2D, faces in 3D) in cells adjacent to cells with a fixed number, i or F, respectively, of topological elements. These quantities are m_i and m_F, respectively. It is more convenient to obtain directly the total number of elements in the neighbours, i.e. the quantities im_i and Fm_F.

4.3.1. Two-dimensional networks. Consider a cell 0 of length L with i or F elements. This cell is in a column 0 and has two adjacent cells in column 0 (base adjacent cells). There are also laterally adjacent cells in the columns that are adjacent to column 0. The number of Poisson points in these columns that fall within the cell is n_k for column k. The corresponding number of laterally adjacent cells is then $n_k + 1$ and the cover length is L_{ck}.

In the 2D laminated Poisson network there are two adjacent columns with n_1, n_2 points inside cell 0, with $I = n_1 + n_2$ and $i = I + 4$ (see figure 1(c)). The number of edges in the $(n_k + 1)$ cells of column k is $4(n_k + 1)$ plus the additional edges that result from contact with cells in column 0 (type 1 edges) and from contact with cells in the following columns which are second neighbour columns relative to column 0 (type 2 edges). The expected number of additional edges of type 1 is 2×2 (contact with the two bases of cell 0) plus the expected number of Poisson points in column 0 in the uncovered lengths $L_{c_1} - L$ and $L_{c_2} - L$ (edges e and e' in figure 1(c)). Each of these expected numbers is 2 (equation (19)). The number of additional edges of type 2 is $\langle L_{c_1} \rangle + \langle L_{c_2} \rangle$ each of which is, from (19) and (12), equal to $2 + (I + 1)/3$. Finally, the expected number of edges in each base adjacent cell is 6, since there is no correlation between the values of i of cells in the same column.

The final result is obtained by summing all these contributions, each of which is written between square brackets:

$$\begin{aligned} im_i &= [4I + 8] + [2 \times 2] + [2 \times 2] + [2(2 + \tfrac{1}{3}(I + 1))] + [2 \times 6] \\ &= (4 + \tfrac{2}{3})I + 32 + \tfrac{2}{3} \end{aligned} \tag{43}$$

or, in terms of $i = I + 4$,

$$im_i = 14 + \tfrac{14}{3}i. \tag{44}$$

The Aboav–Weaire law is exactly followed. The value of a in equation (1) is, since $\mu_2 = 6$,

$$a = 4/3.$$

4.3.2. Three-dimensional networks. We now treat the 3D laminated Poisson networks based on an arbitrary planar network for which the quantities m_i are assumed to be known. Consider a 3D cell 0 with F faces, height L and a base with i edges. We first find the quantity $(Fm_F)_i$ for fixed i and F. This is the total number of faces in cells adjacent to F, i cells. The number of P-points within cell 0 is n_k for each of the distributions $k (k = 1, \ldots, i)$ in the columns adjacent to 0; the number of laterally adjacent cells is $n_k + 1$. We introduce $I_\ell = \sum n_k$; then

$$F = I_\ell + i + 2 \qquad I_\ell = \sum n_k. \tag{45}$$

The number of lateral walls in column k is m_i^k with average value m_i. Then the total number of cells adjacent to cell 0 in column k is $(n_k + 1)(m_i^k + 2)$ plus the number of additional edges that result from contact with other cells. These can be classified into various types, as before. There are type 1 contacts with the cells in column 0; their number is, for column k, $2 + \langle (L_{c_k} - L) \rangle = 4$. There are type 2 contacts with cells in second-neighbour columns. The number of vertical walls of the adjacent cells that contact second neighbours is $im_i - 3i$, provided that there are no triangular bases, in which case there are $3i$ edges of the first neighbour bases that are not contacted by second neighbours. An example for a cell 0 with $i = 5$ is shown in figure 2(a). The $3i = 15$ edges under consideration are the five edges of cell 0 plus the 2×5 edges connected to the vertices of cell 0. The number of extra edges in each of these walls is, from equations (20) and (13), given by $\langle L_c \rangle = 2 + (I_\ell + 1)/(i+1)$. There are contacts of type 3 (with no equivalent in 2D networks) between pairs of adjacent columns k, j that are adjacent to each other. The total number of such adjacencies is i (the adjacencies marked with a dash in figure 2(a)) and each contributes, on average, with $n_k + n_j + 2 + \langle L_{n0} \rangle$ extra faces, where L_{n0} is the uncovered length for the pair k, j ($\langle L_{n0} \rangle = 2$ from equation (20)). Finally, there are two cells adjacent to 0 at the bases, each with $\bar{F}_i$ faces; $\bar{F}_i$ is given by equation (26). Summing all contributions yields

$$(Fm_F)_i = \sum_{k=1}^{i} (n_k + 1)(m_i^k + 2) + 4i + \left(2 + \frac{I_\ell + 1}{i+1}\right)(im_i - 3i) + 2I_\ell + 4i + 2\bar{F}_i. \tag{46}$$

But $\sum n_k m_i^k = m_i I_\ell$ because all distributions of a given set of n_k among the neighbour columns are equiprobable. The final result is

$$(Fm_F)_i = m_i \left[F\frac{1+2i}{1+i} + i - 2 \right] + F\frac{i+4}{i+1} + 7i - 4 \tag{47}$$

where we replaced I_ℓ by F (equation (45)). This equation is valid for any 3D Poisson network with $i \geqslant 4$. For fixed F, the fraction of cells with i bases is defined by $f_F(i)$ given by (28). Then

$$Fm_F = \sum_i f_F(i)(Fm_F)_i. \tag{48}$$

We apply these general equations to the two types of 3D laminated Poisson networks previously discussed. For the hexagonal Poisson network $i = 6$, $m_i = 6$, $f_6 = 1$, $\bar{F} = 14$ and

$$Fm_F = \tfrac{88}{7}F + 62. \tag{49}$$

Aboav–Weaire law is exact, with (cf equation (3))

$$a = 10/7.$$

For the doubly laminated Poisson network m_i is given by (44), and

$$(Fm_F)_i = F\left[\frac{42 + 110i + 31i^2}{3i(1+i)}\right] + \frac{35i^2 + 2i - 84}{3i}. \tag{50}$$

Fm_F was calculated from (48), (47), (28) and (33) for F in the interval $[6, 250]$. Equation (4) is verified. The results are plotted in figure 7. It is clear that Aboav–Weaire law is only approximate. It is apparently exact in the limit of very large F. The best fit straight line can be calculated for a given interval of F and two values of a can be derived from it (see equation (3)). For the interval $[6, F_{max}]$ the two values of a (a_S from the slope, a_0 from the intersection) are as follows: $F_{max} = 20$, $a_S = 2.10$, $a_0 = 1.84$; $F_{max} = 60$, $a_S = 0.16$, $a_0 = 0.68$; $F_{max} = 100$, $a_S = 0.94$, $a_0 = 2.34$. The value of a is therefore strongly dependent on the interval of F to which equation (3) is fitted. A better approximate fitting relation for F in the interval $[6, 250]$ is

$$Fm_F = 337.30 + 11.37F - \frac{9822.0}{F + 30.73} \tag{51}$$

which gives a maximum error of $\sim 2.5\%$. No special meaning should be given to equation (51); other approximate, eventually more complicated, relations between Fm_F and F could be used.

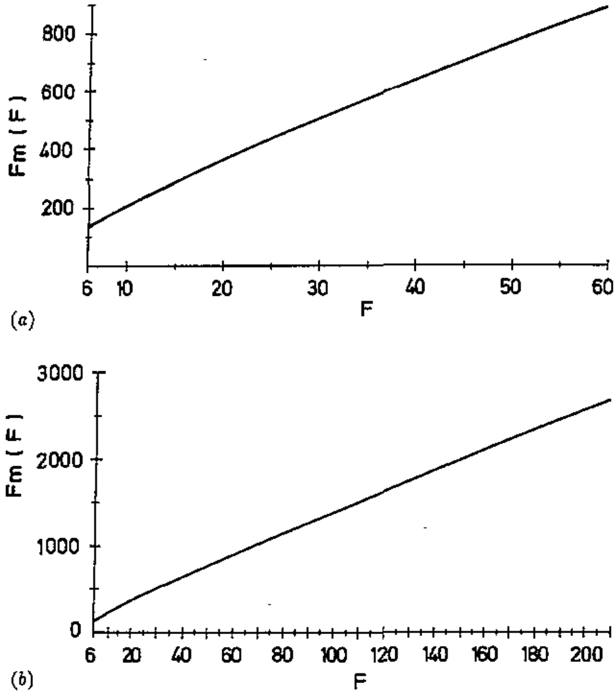

Figure 7. Plots of Fm_F as a function of F for the 3D doubly laminated Poisson network. (a) For $F < 60$. (b) For $F < 200$. The plot tends to a straight line at large F but deviates at low F. The Aboav–Weaire law is not exact.

5. Summary

New types of 2D and 3D random networks (with, respectively, trivalent and tetravalent vertices) were defined and analysed for various topological and geometrical properties, with particular emphasis on the applicability of the Aboav–Weaire and Lewis laws. The networks are of a columnar nature, similar to those shown by some natural cellular materials, such as cork, which are formed by prismatic cells arranged in parallel columns with staggered bases (Natividade 1938, Dormer, 1980). The cells in a column are obtained by a partition (or lamination) which, in the networks analysed, is based on Poisson distributions of given density (Poisson laminated networks). Other laminated networks can be constructed, based on other point distributions (e.g. distributions leading to equiprobable segment lengths in a given interval), eventually with different densities in different columns.

Each Poisson laminated network was analysed for the distribution of the number of topological elements (edges and faces, respectively in 2D and 3D networks), for the average size (area, volume) of cells with fixed number of elements (to test Lewis law) and for the quantities m_i and m_F related to the neighbour correlations (to test Aboav–Weaire law). The 2D Poisson laminated network is remarkable in that both the Lewis and Aboav–Weaire laws (with $a = 4/3$) are exactly followed. This is, to our knowledge, the first example of a 2D network with an infinite interval of i in which these laws are obeyed exactly.

The 3D laminated networks all have $\bar{F} = 14$, no matter on which 2D network they are based. The two laws are exact when the 2D base network is the hexagonal network, the parameter a in equation (3) being $a = 10/7$. When the base network is the 2D Poisson laminated network, both laws are only approximate, but seem to be exact in the limit of large number of topological elements. In such cases, the parameter a in Aboav–Weaire equation (3) depends on the interval of F to which the equation is fitted and no special significance should be attributed to it.

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Appendix

The following identities, valid for any a in [0,1], were used in the calculations of various sums:

$$\sum_{\alpha=0}^{\infty} a^\alpha = \frac{1}{1-a} \qquad \sum_{\alpha=0}^{\infty} \alpha a^\alpha = \frac{a}{(1-a)^2} \qquad \sum_{\alpha=0}^{\infty} \alpha^2 a^\alpha = \frac{a(1+a)}{(1-a)^3}.$$

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